Connes' interpretation of the Standard Model and massive neutrinos

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Massive neutrinos can be accommodated into the noncommutative geometry reinterpretation of the Standard Model.

1. Introduction

The Connes–Lott noncommutative geometry (NCG) approach to fundamental interactions [1, 2] gives a constrained version of the Standard Model (SM). Recently, it has been reported again that neutrinos may have nonvanishing mass [3]. In this letter I outline the accommodation of massive neutrinos into the NCG scheme.

In the Connes–Lott approach, external and internal degrees of freedom of elementary particles are on the same footing. Noncommutative spacetime $\mathcal{A}$ is the product of (the algebra of functions on) the ordinary spacetime M and the algebra of internal degrees of freedom. The necessary mathematical technology to deal with noncommutative manifolds is by now well established [4]. Chirality implies that the internal algebra splits in the direct sum of two subalgebras. A second algebra $\mathcal{B}$, in some sense dual to the first, describes color. The noncommutative spacetime is represented in the fermion Hilbert space:

$$\begin{aligned}\mathcal{H} &= \mathcal{H}_\ell \oplus \mathcal{H}_q \\ &:= L^2(\mathcal{S}_M) \otimes \begin{pmatrix} \mathbb{C}_{e;R} \otimes \mathbb{C}^{N_F} \\ \mathbb{C}^2_{e,\nu;L} \otimes \mathbb{C}^{N_F} \end{pmatrix} \oplus L^2(\mathcal{S}_M) \otimes \begin{pmatrix} (\mathbb{C}_{d;R} \oplus \mathbb{C}_{u;R}) \otimes \mathbb{C}^{N_F} \otimes \mathbb{C}^{N_c} \\ \mathbb{C}^2_{d',u;L} \otimes \mathbb{C}^{N_F} \otimes \mathbb{C}^{N_c} \end{pmatrix}\end{aligned} \tag{1.1}$$

where $\mathcal{S}_M$ denotes the spinor bundle and N_F, N_c respectively the number of particle families and color degrees of freedom.

A generalized Dirac operator also lives on $\mathcal{H}$. One economically obtains the Connes–Lott Lagrangian as a pure, QCD-like, Yang–Mills theory:

$$\mathcal{L} = -\tfrac{1}{4}(F_{NC} \mid F_{NC}) + \langle \Psi \mid D\!\!\!/\, \Psi \rangle,$$

where F_{NC} denotes the NCG gauge field and $D\!\!\!/ = \partial\!\!\!/ + A\!\!\!/_{NC}$ is the generalized Dirac operator, twisted by the NCG gauge potential $A\!\!\!/_{NC}$.

Besides the general mathematical setting, the only inputs of the theory are the Yukawa coupling constants and Kobayashi–Maskawa parameters (YKM constants for short), which enter the definition of the generalized Dirac operator. The Higgs field emerges as the gauge field associated to chirality [5] and the Yukawa terms appear as we apply the minimal coupling prescription with this new gauge potential.

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The purpose of this paper is to explore the consequences of replacing $\mathcal{H}_\ell$ in (1.1) by

$$\tilde{\mathcal{H}}_\ell := L^2(\mathcal{S}_M) \otimes \begin{pmatrix} (\mathbb{C}_{e;R} \oplus \mathbb{C}_{\nu;R}) \otimes \mathbb{C}^{N_F} \\ \mathbb{C}^2_{e,\nu';L} \otimes \mathbb{C}^{N_F} \end{pmatrix}.$$

I take for granted that there is family mixing among leptons [6, 7].

2. Discussion

Computations of Yang–Mills functionals in NCG are by now routine. I refer to our paper [8] for a treatment from first principles. In that paper, however, an important nonlinearity arising in the combination of the quarks and lepton sectors was overlooked. Correct results plus the first identification of Connes–Lott Lagrangian with a constrained SM one appeared in [9]. See also [10–12]. The first step of any NCG computation is to determine the connection 1-forms or gauge potentials. Take the finite part of $\mathcal{A}$ equal to $\mathbb{C} \oplus \mathbb{H}$ and the finite part of $\mathcal{B}$ to $\mathbb{C} \oplus M_3(\mathbb{C})$, as usual. Call π, σ the respective faithful representations on fermion space: $\pi_q(l, q) = (l, \bar{l}, q) \otimes 1_3$ on the quark sector and $\pi_\ell(l, q) = (l, q)$ on the lepton sector. The gauge potentials $\not{A}_\alpha$, $\not{A}_\beta$ are the representations of the unitary connections associated to $\mathcal{A}$ and $\mathcal{B}$, respectively. As such, the theory is $U(1) \times SU(2)_L \times U(1) \times U(3)$ invariant. The reduction to the physical gauge group $SU(2)_L \times U(1)_Y \times SU(3)_c$ is effected by the prescription that the"biconnection" $\not{A}_\alpha + \not{A}_\beta$ be traceless on each chiral sector. For the diagonal part, we schematically had:

$$\not{A}_\alpha + \not{A}_\beta = \begin{array}{c c} & \begin{array}{ccc} R & R & L \end{array} \\ \begin{array}{c} R \\ R \\ L \end{array} & \begin{pmatrix} b + a & & \\ & 0 & \\ & & b + \operatorname{tr} V \end{pmatrix} \end{array} \oplus \begin{array}{c c} & \begin{array}{ccc} R & R & L \end{array} \\ \begin{array}{c} R \\ R \\ L \end{array} & \begin{pmatrix} a + \operatorname{tr} J & & \\ & -a + \operatorname{tr} J & \\ & & \operatorname{tr} J \end{pmatrix} \end{array}.$$

Here a, b, V, J are skewhermitian 1-forms with values in $\mathbb{C}$, $\mathbb{C}$, $\mathbb{H}$ and $M_3(\mathbb{C})$, respectively. Actually $V^* = -V$ means that V is a zero-trace quaternion, so the gauge reduction conditions $\operatorname{Tr}_{\mathcal{H}_R}(\not{A}_\alpha + \not{A}_\beta) = 0$, $\operatorname{Tr}_{\mathcal{H}_L}(\not{A}_\alpha + \not{A}_\beta) = 0$ give $a = b = -\operatorname{tr} J$. It is easily seen that this leads to the correct hypercharge assignments. That was an early triumph of the Connes–Lott approach.

Massive neutrinos do not fit in the original Connes' scheme for leptons [13], as they would give rise to a $U(2) \times U(2)$ gauge group. We can represent ν_R's like u_R type quarks in the current scheme. For each massive neutrino, we have to replace the 0 in the second row of the previous formula by $b - a$. Let us assume that N_1 of the neutrino species are massive and that N_2, with $N_1 + N_2 = N_F$, are massless. The same prescription as before now gives $b = -\operatorname{tr} J$ and then $(N_F + N_1)b + N_2 a + 2N_F \operatorname{tr} J = 0$, leading still to $a = b = -\operatorname{tr} J$ if $N_2 \neq 0$. Then we arrive at the same hypercharge assignments plus zero hypercharge —as desired— for the extant right neutrinos. This is perhaps the more aesthetically pleasing situation, from the standpoint of noncommutative geometry. If all three neutrinos are massive, however, one still gets $b = -\operatorname{tr} J$, but a remains free. This is to say, one of the extra $U(1)$ fields refuses to collapse and the hypercharges are indeterminate. The simpler solution to the problem is to impose $a = b$ nevertheless. This is natural if one wants to

regard the massless case as the limit of the massive case. Such a choice yields again the correct hypercharges of the SM and also automatically $Y(\nu_R) = 0$. I do not claim that the method adopted in this paper is the only way of fitting massive neutrinos in the Connes–Lott formalism; but it is clearly the simplest one, running in close parallel to the current treatment.

When all computations are done, the boson part of the Connes–Lott Lagrangian with massive neutrinos is of the form $\mathcal{L}_2 + \mathcal{L}_1 + \mathcal{L}_0$ with

$$\begin{aligned} \mathcal{L}_2 &= -BF_{\mu\nu}F^{\mu\nu} - \tfrac{1}{4}CH^a_{\mu\nu}H_a^{\mu\nu} - AG^a_{\mu\nu}G_a^{\mu\nu}, \\ \mathcal{L}_1 &= 2L\,(D_\mu\Phi\, D^\mu\Phi), \\ \mathcal{L}_0 &= K\,(\|\Phi_1\|^2 + \|\Phi_2\|^2 - 1)^2, \end{aligned} \tag{2.1}$$

with some coefficients A, B, C, L, K. Here F, H, G respectively denote the $U(1)$, $SU(2)$ and $SU(3)$ gauge fields and $\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}$ is a Higgs doublet. The fermion part is the standard one, containing the Yukawa terms for massive neutrinos.

3. Other issues

I look at the matter of the constraints suggested by NCG next. The first NCG treatments of the SM took simply $F_{NC} = F_\alpha + F_\beta$, where F_α, F_β of course denote the curvatures associated to the connections $\not{A}_\alpha$, $\not{A}_\beta$. This leads to the "predictions" $\mathrm{M_{top}} = 2\,\mathrm{M}_W$ and $\mathrm{M_{Higgs}} = 3.14\,\mathrm{M}_W$ (sitting nicely in the right ballpark) and to values for α_3 and $\sin^2\theta_W$ in clear disagreement with experiment. Kastler and Schücker [9] then allowed themselves the freedom of combining the quark and lepton sectors of $F_\alpha + F_\beta$ with different coefficients c_ℓ and c_q; this leads to more or less acceptable α_3 and $\sin^2\theta_W$ values only in the "lepton dominance" limit $c_q/c_\ell \to 0$, for which there is a huge predicted top mass. This was the end of the "fundamentalist" period. Meanwhile, the "revisionist attitude" was being conducted by Connes himself, hoping perhaps to put an end to controversy. He reckoned that π, σ were not irreducible representations and that replacing $(F_\alpha + F_\beta \mid F_\alpha + F_\beta)$ by a more general scalar product

$$\lambda_\alpha(F_\alpha \mid F_\alpha) + \lambda_\beta(F_\beta \mid F_\beta), \tag{3.1}$$

with λ_α a positive matrix in the commutant of $\{\mathcal{A}, \not{D}\}$ and similarly λ_β in the commutant of $\{\mathcal{B}, \not{D}\}$, he would gain a lot of free parameters for the theory. He concluded that, while NCG seems to give some "preferred" values for the yet to be discovered top and Higgs particle masses, in its more general form gives rise to no constraints whatsoever.

However, quite recently, a rather exhaustive analysis [11, 12] by Kastler and Schücker has shown that (i) although the commutant of $\{\mathcal{B}, \not{D}\}$ is quite big, it intervenes only through a couple of parameters c'_ℓ, c'_q, entering just the determination of the coupling constants of the external gauge fields; (ii) thus one can conveniently shift the "lepton dominance" to the $\not{A}_\beta$ sector, accommodating the experimental values of the strong coupling constant and Weinberg's angle; and (iii) some restrictions in the Higgs sector remain: there are the "absolute" (i.e., independent of the chosen scalar product) constraints

$$\mathrm{M_{top}} \geq \sqrt{3}\,\mathrm{M}_W; \quad \sqrt{7/3}\,\mathrm{M_{top}} \leq \mathrm{M_{Higgs}} \leq \sqrt{3}\,\mathrm{M_{top}}, \tag{3.2}$$

where the approximation of neglecting all fermion masses except for the top is made. Moreover, once the top mass is assumed known, one could after all give a unique value for the Higgs mass. The latter constraints depend solely on the λ_α parameters. Actually, $\{\pi(\mathcal{A}), \not{D}\}$ generate the full flavor part of $\mathcal{H}_q$, so that from the quark sector we get only the old parameter c_q. With massless neutrinos we get a parameter for each family: schematically $\lambda_\alpha = (c_e, c_\mu, c_\tau; c_q)$. However, only the sum $c_e + c_\mu + c_\tau =: 3c_\ell$ plays any practical role. With massive neutrinos, the action of $\{\pi(\mathcal{A}), \not{D}\}$ on $\mathcal{H}_\ell$ is obviously irreducible and we are back to the couple c_ℓ, c_q.

Formula (3.1) appears to overlook the fact that F_α and F_β are not orthogonal. Therefore the more general scalar product should involve a $\lambda_{\alpha\beta}(F_\alpha \,|\, F_\beta)$ term, which is perfectly invariant under the physical gauge group. This term modifies only the abelian part of the NCG gauge field. Noninclusion of it, however, leads to the weird consequence that the new analysis by Kastler and Schücker does not reduce to their old in any limit. Massive neutrinos modify the situation again: then F_α and F_β are orthogonal (not unrelated to the fact that we were not able to determine the abelian part of $\not{A}_\alpha$ without a supplementary condition) and both kind of analysis become compatible.

With N_F and N_c equal to 3, and $N_1 = N_F$, I obtain for the external field coefficients in (2.1)

$$A = 3c'_q; \quad B = 3c_\ell + 6c'_\ell + 9c_q + 2c'_q; \quad C = 3c_\ell + 9c_q,$$

and for the internal ones

$$L = c_\ell \operatorname{tr}(g_\nu^\dagger g_\nu + g_e^\dagger g_e) + 3c_q \operatorname{tr}(g_d^\dagger g_d + g_u^\dagger g_u)$$

and

$$\begin{aligned} K = {}& \tfrac{3}{2} c_\ell \operatorname{tr}((g_\nu^\dagger g_\nu)^2 + (g_e^\dagger g_e)^2) + c_\ell \operatorname{tr}(g_\nu^\dagger g_\nu g_e^\dagger g_e) \\ & + \tfrac{9}{2} c_q \operatorname{tr}((g_d^\dagger g_d)^2 + (g_u^\dagger g_u)^2) + 3c_q \operatorname{tr}(g_d^\dagger g_d g_u^\dagger g_u) - \frac{L^2}{3c_\ell + 9c_q} \end{aligned}$$

in terms of the YKM constants and four strictly positive constants $c_\ell, c_q, c'_\ell, c'_q$, unknown a priori. Besides an overall multiplication constant, we are left with three parameters. I choose the useful $x := \dfrac{c_\ell - c_q}{c_\ell + c_q}$, as in [9, 12]; $x' := \dfrac{c'_\ell - c'_q}{c'_\ell + c'_q}$ and $u := \dfrac{c'_\ell + c'_q}{c_\ell + c_q}$. One has $-1 \leq x \leq 1$; $-1 \leq x' \leq 1$; $0 \leq u < \infty$. Identification to the standard Lagrangian gives the constraints. I list the ones which are not the same as the counterparts with massless neutrinos:

$$\sin^2\theta_W = \frac{C}{B + C} = \frac{1}{2 + u(8 + 4x')/(12 - 6x)}. \tag{3.3a}$$

In particular, $\sin^2\theta_W \leq 0.5$ is an "absolute" constraint. For the reasons given at the beginning of this Section, that formula cannot be directly compared to the corresponding one in [12]. It does reduce to the formula in [9] for the all-massless case when $x' = x$ and $u = 1$. Also:

$$\frac{\mathrm{M_{Higgs}}}{\mathrm{M_{top}}} = \sqrt{\frac{2K}{L}} = \sqrt{3 - \frac{1 - x}{2 - x}} = \sqrt{3 - \frac{2\,\mathrm{M_W^2}}{\mathrm{M_{top}^2}}}. \tag{3.2d}$$

This function increases slowly with x. It is much simpler than its counterparts with massless neutrinos, which are unbearably ugly (see [9, 12] for the all-massless case). It gives slightly higher masses for the Higgs particle than the equivalent ones with massless neutrinos. For instance, the NCG value $x = 0$ here yields $\mathrm{M}_{\mathrm{Higgs}} = 253.7$ GeV instead of $\mathrm{M}_{\mathrm{Higgs}} = 251.7$ GeV [5, 12]. The "absolute" constraints (3.2) would still hold.

4. Conclusions

Noncommutative geometry gives no clue about the wide spectrum of fermion masses, but it apparently points to both the intermediate boson and the Higgs masses being of the same order as the highest fermion masses. The constraint $\mathrm{M}_W \leq \mathrm{M}_{\mathrm{top}}/\sqrt{N_F}$ is suggestive. Of course we have remained at the classical level throughout. At present, there seems to be no compelling reason to adopt Connes' relations on-shell. One can take the point of view that any constraints can be meaningfully imposed only in a renormalization group invariant way. This I showed, together with E. Alvarez and C. P. Martín, to be impossible, if one performs the quantization in ordinary quantum field theory [14, 15]. It occurred to several people that quantization should be performed with due account of the (elusive) symmetry associated to the interpretation of the Higgs particle as another gauge boson; but nobody seems to know how to go about it.

Connes–Lott models are much more rigid than ordinary Yang–Mills–Higgs ones. This is reflected in the fact that "most" of the latter cannot be obtained from the former [16]; it is altogether remarkable that the SM, *without or with* some right-handed neutrinos is "one of the few" that can. As well, the resulting constraints reflect that rigidity, which is welcome if it were to lead to useful physical predictions. The consequences of assuming a novanishing neutrino mass are partly discontinuous in the mass variable; the precise values of the YKM constants for neutrinos are unimportant in our context.

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